MODULE - 4
GENERATING FUNCTIONS AND RECURRENCE RELATIONS

Syllabus

* Generating Function
* Definition and Examples
* Calculation techniques,
* Exponential generating functions
* First order linear recurrence relation with constant co-effts.
* Homogeneous, Non homogeneous solutions.
* Second order linear recurrence relation with constant co-efficients
* Homogeneous non homogeneous solutions.

Generating functions

Defn:-

Let $a_0, a_1, a_2 \ldots$ be a sequence of real numbers. The function $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{i=0}^{\infty} a_i x^i$ is called the GENERATING FUNCTION of the given sequence. (It is a power series in x)

Note:-

Given a sequence, we can easily obtain its generating functions & from the generating function we can reproduce the sequence.

Vimp **Some Results**

1) $(1+x)^n = \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \ldots + \binom{n}{n} x^n$ $\quad \boxed{n \in Z^+}$

$\therefore (1+x)^n$ is the generating function for

$${}^nC_0, {}^nC_1, {}^nC_2, \ldots {}^nC_n$$

OR

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \ldots$$

$$(1-x)^n = 1 - nx + \frac{n(n-1)x^2}{2!} - \frac{n(n-1)(n-2)}{3!} x^3 + \ldots$$

2) $(1+x)^{-1} = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$

ie $(1+x)^{-1}$ is the generating fn. for $1, -1, 1, -1, \ldots$

3) $(1+x)^{-2} = \frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

It is the generating function for $1, -2, 3, -4, \ldots$

4) $(1-x)^{-1} = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

generates the sequence $1, 1, 1, \ldots$

5) $(1-x)^{-2} = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

generates the sequence $1, 2, 3, 4, \ldots$

6) $(1+x)^{-n} = \binom{-n}{0} + \binom{-n}{1}x + \binom{-n}{2}x^2 + \cdots$

$$= 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \cdots$$

7) $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \cdots$

Then all +ve

1) Find the co-efficient of x^5 in $(1-2x)^{-7}$

Ans:- using result ⑦ in page 2 . put $2x = y$,

$$(1-y)^{-n} = 1 + ny + \frac{n(n+1)}{2!}y^2 + \frac{n(n+1)(n+2)}{3!}y^3 + \cdots$$

$$\boxed{n = 7, \quad y = 2x}$$

$$= 1 + 7(2x) + \frac{7\cdot 8}{2!}(2x)^2 + \frac{7\cdot 8\cdot 9}{3!}(2x)^3 + \frac{7\cdot 8\cdot 9\cdot 10}{4!}(2x)^4$$

$$+ \boxed{\frac{7\cdot 8\cdot 9\cdot 10\cdot 11}{5!}(2x)^5} + \cdots$$

$$\therefore \text{Co-efficient of } x^5 = \frac{7\times 8\times 9\times 10\times 11}{5!}\times 2^5$$

$$= 14,784$$

2) Determine the co-efficient of x^{15} in $f(x)$,

$$f(x) = (x^2 + x^3 + x^4 + \cdots)^4$$

Ans $f(x) = (x^2 + x^3 + x^4 + \cdots)^4$

$$= [x^2(1 + x + x^2 + \cdots)]^4$$

$$= x^8(1 + x + x^2 + \cdots)^4$$

$$= x^8[(1-x)^{-1}]^4 = \frac{x^8}{(1-x)^4} = x^8(1-x)^{-4}$$

using 7th result in pge 2

$$= x^8\left[1 + 4x + \frac{4\cdot 5}{2!}x^2 + \left(\frac{4\cdot 5\cdot 6}{3!}\right)x^3 + \left(\frac{4\cdot 5\cdot 6\cdot 7}{4!}\right)x^4 + \left(\frac{4\cdot 5\cdot 6\cdot 7\cdot 8}{5!}\right)x^5\right.$$

$$+ \cdots$$

$\therefore$ Co-efficient of $x^{15} = \frac{4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10}{7!} = \underline{\underline{120}}$

3) Find the co-efficient of x^8 in $\frac{1}{(x-3)(x-2)^2}$

Ans- Apply partial fraction.

$$\frac{1}{(x-3)(x-2)^2} = \frac{A}{x-3} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$

$$1 = A(x-2)^2 + B(x-3)(x-2) + C(x-3)$$

$x=2$
$1 = C(-1) \therefore \underline{\underline{C=-1}}$

$x=3$
$\underline{\underline{1=A}}$

$x=0$
$1 = 4A + 6B - 3C$
$1 = 4 + 6B + 3$
$1 = 7 + 6B$
$-6 = 6B \quad \therefore \underline{\underline{B=-1}}$

$$\therefore \frac{1}{(x-3)(x-2)^2} = \frac{1}{(x-3)} - \frac{1}{(x-2)} - \frac{1}{(x-2)^2}$$

$$= \frac{1}{-3\left(1-\frac{x}{3}\right)} + \frac{1}{2\left(1-\frac{x}{2}\right)} - \frac{1}{4\left(1-\frac{x}{2}\right)^2}$$

$$= -\frac{1}{3}\left(1-\frac{x}{3}\right)^{-1} + \frac{1}{2}\left(1-\frac{x}{2}\right)^{-1} - \frac{1}{4}\left(1-\frac{x}{2}\right)^{-2}$$

$$= -\frac{1}{3}\left[1+\left(\frac{x}{3}\right)+\left(\frac{x}{3}\right)^2+\cdots+\left(\frac{x}{3}\right)^8+\cdots\right] +$$
$$\frac{1}{2}\left[1+\left(\frac{x}{2}\right)+\left(\frac{x}{2}\right)^2+\cdots+\left(\frac{x}{2}\right)^8+\cdots\right] +$$
$$-\frac{1}{4}\left[1+2\left(\frac{x}{2}\right)+3\left(\frac{x}{2}\right)^2+\cdots+9\left(\frac{x}{2}\right)^8+\cdots\right]$$

$\therefore$ Co-efficient of $x^8 = \left[-\frac{1}{3}\left(\frac{1}{3}\right)^8 + \frac{1}{2}\left(\frac{1}{2}\right)^8 - \frac{9}{4}\left(\frac{1}{2}\right)^8\right] = \underline{\left(\frac{-1}{3}\right)^9 + -7\left(\frac{1}{2}\right)^{10}}$

4. Determine the constant (co-efficient of x^0) in $\left(4x^3 - \frac{5}{x}\right)^{16}$

Ans:- $\left(4x^3 - \frac{5}{x}\right)^{16} = \left[\frac{4x^4 - 5}{x}\right]^{16}$

$$= \frac{1}{x^{16}} (4x^4 - 5)^{16}$$

$$= \frac{1}{x^{16}} (-5)^{16} \left[1 - \frac{4}{5}x^4\right]^{16}$$

$$= (-5)^{16} \cdot \frac{1}{x^{16}} \left[1 - 16\left(+\frac{4}{5}x^4\right) + \frac{16 \cdot 15}{2!}\left(+\frac{4}{5}x^4\right)^2 - \frac{16 \cdot 15 \cdot 14}{3!}\left(+\frac{4}{5}x^4\right)^3 + \frac{16 \cdot 15 \cdot 14 \cdot 13}{4!} + \left(\frac{4}{5}\right)^4 (x^4)^4 + \cdots\right]$$

$\therefore$ Co-efft of constant $= (-5)^{16} \times \left(\frac{4}{5}\right)^4 \times \frac{16 \cdot 15 \cdot 14 \cdot 13}{1 \cdot 2 \cdot 3 \cdot 4}$

$$= -5^{12} \cdot (-4)^4 \times {}^{16}C_{12}$$

H.W.

5. Find the co-efficient of x^{60} in $(x^8+x^9+x^{10}+\cdots)^7$

6. Determine the sequence generated by $(1-4x)^{-\frac{1}{2}}$.

We have

$$(1-x)^{-n} = 1+nx+\frac{n(n+1)}{2!}x^2+\frac{n(n+1)(n+2)}{3!}x^3+\cdots\cdots$$

$$(1-4x)^{-\frac{1}{2}} = 1+\frac{1}{2}(4x)+\frac{\frac{1}{2}(\frac{1}{2}+1)}{2!}(4x)^2+\frac{\frac{1}{2}(\frac{1}{2}+1)(\frac{1}{2}+2)}{3!}(4x)^3+\cdots$$

$$\cdots\frac{\overbrace{(\frac{1}{2})(\frac{1}{2}+1)(\frac{1}{2}+2)\cdots(\frac{1}{2}+n-1)}^{n \text{ terms}}}{n!}(4x)^n$$

Co-efficient of x^n is $\dfrac{(\frac{1}{2})(\frac{1}{2}+1)(\frac{1}{2}+2)\cdots\cdots[\frac{1}{2}+(n-1)]\cdot 4^n}{n!}$

$$= \frac{(1)(1+2)(1+4)\cdots\cdots(1+2n-2)}{2^n\cdot n!}\cdot 4^n$$

$$= \frac{1\cdot3\cdot5\cdots(2n-3)(2n-1)\,2^n}{n!}$$

$$= \frac{1\cdot3\cdot5\cdots(2n-3)(2n-1)\,2^n}{n!}\times\frac{n!}{n!} \quad \left(\begin{array}{l}\text{Multiply}\\ \text{Nr \& Dr by } n!\end{array}\right)$$

$$= \frac{1\cdot3\cdot5\cdots(2n-3)(2n-1)\,2^n\underbrace{(1\cdot2\cdot3\cdots(n-2)(n-1)n)}_{n \text{ terms}}}{n!\;n!}$$

$$= \frac{1\cdot3\cdot5\cdots(2n-3)(2n-1)\,(2\cdot4\cdot6\cdots(2n-4)(2n-2)\cdot2n)}{n!\;n!}$$

$$= \frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdots(2n-4)(2n-3)(2n-2)(2n-1)(2n)}{n!\;n!}$$

$$= \frac{(2n)!}{n!\,n!} = {}^{2n}C_n = \binom{2n}{n}$$

The sequence is $1, {}^2C_1, {}^4C_2, {}^6C_3, {}^8C_4, \ldots\ldots$

H.W.

7. Compute the co-efficient of x^n in $(x+x^2+\cdots+x^5)(x^2+x^3+\cdots+x^n+\cdots)^3$

8. Find the generating functions for the following sequences.

ⓐ $0, 0, 0, -6, +6, -6, 6 \ldots$

ⓑ $\binom{8}{0}, \binom{8}{1}, \binom{8}{2}, \ldots \binom{8}{8}$

HW

ⓒ $\binom{8}{1}, 2\binom{8}{2}, 3\binom{8}{3}, \ldots 8\binom{8}{8}$

ⓓ $1, -1, 1, -1, 1, -1 \ldots$

ⓔ $1, 0, 1, 0, 1, 0, 1, \ldots$

ⓕ $0, 0, 1, a, a^2, a^3, \ldots$ where $a \neq 0$

Ans: ⓐ $f(x) = 0 \cdot x^0 + 0x^1 + 0x^2 - 6x^3 + 6x^4 - 6x^5 + 6x^6 + \cdots$

$f(x) = -6x^3 + 6x^4 - 6x^5 + 6x^6 + \cdots$ ($-6x^3$ is common for all terms. So take it outside)

$= -6x^3[1 - x + x^2 - x^3 + \cdots]$ result ② in pg 2

$= \underline{-6x^3(1+x)^{-1}} = \dfrac{-6x^3}{1+x}$

ⓑ $f(x) = \binom{8}{0} + \binom{8}{1}x + \binom{8}{2}x^2 + \binom{8}{3}x^3 + \cdots + \binom{8}{8}x^8$

$f(x) = \underline{(1+x)^8}$ from Result ① in pg no: 1

ⓒ $g(x) = ({}^8C_1) + 2({}^8C_2)x + 3({}^8C_3)x^2 + \cdots 8({}^8C_8)x^7$

from the previous problem, $f(x) = {}^8C_0 + ({}^8C_1)x + ({}^8C_2)x^2 + \cdots + ({}^8C_8)x^8$

Differentiate both sides

$f'(x) = {}^8C_1 + 2({}^8C_2)x + 3({}^8C_3)x^2 + \cdots + 8({}^8C_8)x^7$

$= g(x).$

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ie $g(x) = f'(x)$.

$$\boxed{\begin{aligned} f(x) &= (1+x)^8 \quad \text{from (b)} \\ f'(x) &= 8(1+x)^7 \end{aligned}}$$

$\therefore$ The generating function is

$$g(x) = \underline{\underline{8(1+x)^7}}$$

d

e

f

1. $\binom{n-4}{n-7} - \binom{n-9}{n-12}$

Hint:

$$x^7\left(1+x+x^2+x^3+x^4\right)\left(1+x+x^2+\cdots\right)^3$$
$$=x^7\left(\frac{1-x^5}{1-x}\right)\left(\frac{1}{1-x}\right)^3=x^7\left(1-x^5\right)(1-x)^{-4}$$

x^n results from $x^7 \cdot 1 \cdot \binom{n-7+4-1}{n-7}x^{n-7}$ and $x^7(-x^5)\binom{n-12+4-1}{n-12} \times x^{n-12}$

9. Determine the sequence generated by each of the following generating functions.

a) $f(x) = (2x-3)^3$

b) $f(x) = x^4/(1-x)$

H.W c) $f(x) = \frac{1}{(3-x)}$

d) $f(x) = 1/(1-x) + 3x^7 - 11$

H.W e) $f(x) = \frac{1}{1+3x}$

H.W f) $f(x) = \frac{x^3}{1-x^2}$

Ans:- (a) $f(x) = (2x-3)^3$

$$= \left[-3\left(1-\frac{2}{3}x\right)\right]^3 = -27\left(1-\frac{2}{3}x\right)^3$$

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

$$= -27\left[1 - 3\left(\frac{2}{3}x\right) + \frac{3\cdot 2}{2}\left(\frac{2}{3}x\right)^2 - \frac{3\cdot 2\cdot 1}{3!}\left(\frac{2}{3}x\right)^3\right]$$

$$= -27\left[1 - 2x + \frac{2^2}{3}x^2 - \left(\frac{2}{3}\right)^3 x^3\right]$$

$$= \underbrace{-27}_{a_0} + \underbrace{27(2)}_{a_1}x \underbrace{- 27\times\left(\frac{2^2}{3}\right)}_{a_2}x^2 + \underbrace{27\cdot\left(\frac{2}{3}\right)^3}_{a_3}x^3$$

Note:-

$a_0, a_1, a_2, \cdots$ is the sequence

a_0: co-efft of x^0

a_1: co-efft of x^1

a_2: co-efft of x^2

⋮

a_n: co-efft of x^n

⋮

∴ The sequence is $a_0, a_1, a_2, a_3, \ldots$

$-27, 54, -36, 8, 0, 0, 0 \cdots$

(b) $f(x) = \frac{x^4}{1-x} = x^4[1-x]^{-1} = x^4(1+x+x^2+x^3+\cdots)$

$$= x^4 + x^5 + x^6 + x^7 + \cdots$$

$a_0 = 0, a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 1, a_5 = 1, \ldots$

∴ The sequence is $0, 0, 0, 0, 1, 1, 1, \ldots$

d) $f(x) = \frac{1}{1-x} + 3x^7 - 11$

$= (1-x)^{-1} + 3x^7 - 11$

$= (1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \cdots) + 3x^7 - 11$

$= -10 + x + x^2 + x^3 + x^4 + \cdots + 4x^7 + x^8 + \cdots$

$\left.\begin{array}{ll} a_0 = -10 & \\ a_7 = 4 & \\ a_i = 1 & i \neq 0, 7 \end{array}\right\}$ is the sequence

OR $-10, 1, 1, 1, 1, 1, 1, 4, 1, 1, \ldots$

c

e

f

Convolution of the sequences $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$

$c = a * b$.

Whenever a sequence $c_0, c_1, c_2 \ldots$ arise from two generating functions $f(x)$ [for $a_0, a_1, a_2 \ldots$] & $g(x)$ [for $b_0, b_1, b_2, \ldots$],

the sequence $\boxed{c_k = a_0 b_k + a_1 b_{k-1} + a_2 b_{k-2} + \cdots + a_{k-1} b_1 + a_k b_0}$ is called the convolution of $a_0, a_1, a_2 \ldots$ & $b_0, b_1, b_2, \ldots$

$a_0, a_1, a_2, \ldots a_r$

$b_r, b_{r-1}, b_{r-2}, \cdots b_1, b_0$

$$
\boxed{\begin{aligned}
c_0 &= a_0 b_0 \\
c_1 &= a_0 b_1 + a_1 b_0 \\
c_2 &= a_0 b_2 + a_1 b_1 + a_2 b_0 \\
c_3 &= a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 \\
&\vdots
\end{aligned}}
$$

The generating function for $c_0, c_1, c_2, c_3 \ldots$ is given by $\boxed{h(x) = f(x) g(x)}$

$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

$g(x) = b_0 + b_1 x + b_2 x^2 + \cdots$

$h(x) = c_0 + c_1 x + c_2 x^2 + \cdots$

1. Find the first four terms c_0, c_1, c_2 and c_3 of the convolutions for each of the following pairs of sequences.

(a) $a_n = 1 \quad b_n = 1 \quad \forall n \in N$

(b) $a_n = 1 \quad b_n = 2^n$ for all $n \in N$

(c) $a_0 = a_1 = a_2 = a_3 = 1, \quad a_n = 0 \quad n \in N, \ n \neq 0, 1, 2, 3$
$b_n = 1 \quad \forall n \in N$

Ans:- (a) $C_0 = a_0b_0 = 1\times 1 = \underline{1}$
$C_1 = a_0b_1 + a_1b_0 = \underline{2}$
$C_2 = a_0b_2 + a_1b_1 + a_2b_0 = \underline{3}$
$C_3 = a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0 = \underline{4}$

$a_0 = 1$, $b_0 = 1$
$a_1 = 1$, $b_1 = 1$
$a_2 = 1$, $b_2 = 1$
$a_3 = 1$, $b_3 = 1$

(b) $C_0 = a_0b_0 = 1\times 2^0 = \underline{1}$
$C_1 = a_0b_1 + a_1b_0 = 2+1 = \underline{3}$
$C_2 = a_0b_2 + a_1b_1 + a_2b_0 = 4+2+1 = \underline{7}$
$C_3 = a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0 = 8+4+2+1 =$

$a_0 = 1$, $b_0 = 2^0$
$a_1 = 1$, $b_1 = 2^1$
$a_2 = 1$, $b_2 = 2^2$
$a_3 = 1$, $b_3 = 2^3$

(c) $C_0 = 1$
$C_1 = 2$
$C_2 = 3$
$C_3 = 1$

$a_0 = 1$, $b_1 = 1$
$a_1 = 1$, $b_2 = 1$
$a_2 = 1$, $b_3 = 1$
$a_3 = 1$, $b_4 = 1$
$a_4 = 0$
$a_5 = 0$
$\vdots$

2.) Find a formula for the convolution of each of the following pairs of sequences.

(a) $a_n = 1 \quad 0 \le n \le 4, \quad a_n = 0 \quad \forall n \ge 5$
$b_n = n \quad \forall n \in N$

(b) $a_n = (-1)^n, \quad b_n = (-1)^n \quad \forall n \in N.$

Ans:- (a) $a_0 = 1, a_1 = 1, a_2 = 1, a_3 = 1, a_4 = 1, a_5 = 0, a_6 = 0, \ldots$
$b_0 = 0, b_1 = 1, b_2 = 2, b_3 = 3, b_4 = 4, b_5 = 5, b_6 = 6, \ldots$

generating function for a_n and b_n are

$f(x) = 1 + x + x^2 + x^3 + x^4$
$g(x) = 0 + x + 2x^2 + 3x^3 + 4x^4 + \ldots$ respectively.

$\therefore \underbrace{h(x)}_{\text{for } C_n \text{(convolution)}} = f(x)g(x) = (1+x+x^2+x^3+x^4)(x+2x^2+3x^3+4x^4+\ldots)$

This is the gen. function for C_n.

$c_0 = a_0 b_0 = \underline{\underline{0}}$

$c_1 = a_0 b_1 + a_1 b_0 = 1 + 0 = \underline{\underline{1}}$

$c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0 = 2 + 1 + 0 = \underline{\underline{3}}$

$c_3 = a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 = 3 + 2 + 1 + 0 = \underline{\underline{6}}$

$c_4 = 4 + 3 + 2 + 1 + 0 = \underline{\underline{10}}$

$c_5 = 5 + 4 + 3 + 2 + 1 + 0 = 15$

$c_6 = 6 + 5 + 4 + 3 + 2 + 0 + 0 = 20$

$\vdots$

(sum of 5 terms)

$c_n = n + (n-1) + (n-2) + (n-3) + (n-4) \quad \forall\, n \geq 5$

$$\underline{\underline{c_n = 5n - 10 \quad \forall\, n \geq 5}}$$

(b) $a_n = (-1)^n \qquad a_n = 1, -1, 1, -1, 1, -1, \ldots$

$b_n = (-1)^n \qquad b_n = 1, -1, 1, -1, 1, -1, \ldots$

generating fn of $a_n = 1 - x + x^2 - x^3 + \cdots = (1+x)^{-1} = f(x)$

" " " $b_n = 1 - x + x^2 - x^3 + \cdots = (1+x)^{-1} = g(x)$

$\therefore$ generating fn for convolution $c_n = h(x) = f(x) \cdot g(x)$

$$= (1+x)^{-1}(1+x)^{-1}$$

$$= \frac{1}{(1+x)} \cdot \frac{1}{1+x} = \frac{1}{(1+x)^2}$$

$$h(x) = (1+x)^{-2}$$

$h(x) = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots$

$c_0 = 1, \; c_1 = -2, \; c_2 = 3, \; c_3 = -4 \; \ldots$

$$\therefore \underline{\underline{c_n = (-1)^n (n+1)}} \quad \forall\, n \in N$$

Questions

1. Obtain the generating functions for the sequence.

$3^0, 3^1, 3^2, 3^3, \dots 3^r, \dots$

Ans:- $f(x) = 3^0 + 3^1x + 3^2x^2 + 3^3x^3 + \dots$

$= 1 + (3x) + (3x)^2 + (3x)^3 + \dots$

We have
$1 + x + x^2 + x^3 + \dots = (1-x)^{-1}$

$= (1-3x)^{-1} = \underline{\underline{\frac{1}{1-3x}}}$ is the generating fn.

II Exponential Generating functions.

Defn:- For a sequence of real numbers $a_0, a_1, a_2, a_3 \ldots$

$$\boxed{f(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}}$$

is called the exponential generating function for the given sequence

Results (Important exponential generating functions)

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots$ generates $1, 1, 1, 1, \ldots$

$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} \cdots$ generates $1, -1, 1, -1, 1, -1, \ldots$

$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$ generates $1, 0, 1, 0, 1, 0, \ldots$

$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$ generates $0, 1, 0, 1, 0, 1, 0, 1 \ldots$

Note:- e^x is the ordinary generating function for the sequence $1, 1, \frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}, \ldots$

$$\therefore e^x = \overset{a_0}{(1)} + \overset{a_1}{(1)} x + \overset{a_2}{\left(\frac{1}{2!}\right)} x^2 + \overset{a_3}{\left(\frac{1}{3!}\right)} x^3 + \cdots$$

1) Find the exponential generating function for each of the following sequences.

H.W (a) $1, -1, 1, -1, 1, -1, \ldots\ldots$

(b) $1, 2, 2^2, 2^3, 2^4, \ldots\ldots$

(c) $1, -a, a^2, -a^3, a^4, \ldots\ldots$ $a \in R$.

(d) $1, a^2, a^4, a^6, \ldots\ldots$ $a \in R$

H.W (e) $a, a^3, a^5, a^7, \ldots\ldots$ $a \in R$

(f) $0, 1, 2(2), 3(2^2), 4(2^3), \ldots$

Ans:- (a)

(b) $f(x) = 1 + 2 \cdot \frac{x}{1!} + 2^2 \cdot \frac{x^2}{2!} + 2^3 \cdot \frac{x^3}{3!} + 2^4 \cdot \frac{x^4}{4!} + \ldots\ldots$

$$= 1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \ldots$$

$$= e^{2x}$$

(c) $f(x) = 1 - \frac{ax}{1!} + \frac{a^2x^2}{2!} - \frac{a^3x^3}{3!} + \frac{a^4x^4}{4!} + \ldots\ldots$

$$= 1 - \frac{ax}{1!} + \frac{(ax)^2}{2!} - \frac{(ax)^3}{3!} + \frac{(ax)^4}{4!} + \ldots$$

$$= e^{-ax}$$

(d) $f(x) = 1 + a^2 \frac{x}{1!} + a^4 \cdot \frac{x^2}{2!} + a^6 \cdot \frac{x^3}{3!} + \ldots\ldots$

$$= 1 + \frac{a^2x}{1!} + \frac{(a^2x)^2}{2!} + \frac{(a^2x)^3}{3!} + \ldots\ldots$$

$$= e^{a^2x}$$

(e)

(f) $f(x) = 0 + x + 2(2)\frac{x^2}{2!} + 3(2^2)\cdot\frac{x^3}{3!} + 4(2^3)\frac{x^4}{4!} + \cdots$

$= x\left[1 + 2^2\cdot\frac{x^1}{2!} + 3(2^2)\cdot\frac{x^2}{3!} + 4(2^3)\frac{x^3}{4!} + \cdots\right]$

$= x\left[1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots\right]$

$= \underline{\underline{x\cdot e^{2x}}}$

2) Determine the sequences generated by each of the following exponential generating functions. $f(x) =$

ⓐ $5e^{5x}$ H.W ⓑ $3e^{3x}$

ⓒ $7\cdot e^{8x} - 4e^{3x}$ H.W ⓓ $6e^{5x} - 3e^{2x}$

ⓔ $2e^{x} + 3x^2$ H.W ⓕ $e^{x} + x^2$

ⓖ $f(x) = e^{3x} - 28x^3 - 6x^2 + 9x$ H.W ⓗ $f(x) = e^{2x} - 3x^3 + 5x^2 + 7x$

Ans:- ⓐ $f(x) = 5\left[1 + 5x + \frac{(5x)^2}{2!} + \frac{(5x)^3}{3!} + \cdots\right]$

$\therefore$ sequence is $\underline{5, 5^2, 5^3, 5^4, \cdots}$

ⓒ $f(x) = 7\cdot e^{8x} - 4e^{3x}$

$= 7\cdot\left[1 + 8x + \frac{(8x)^2}{2!} + \frac{(8x)^3}{3!} + \cdots\right] - 4\left[1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \cdots\right]$

$= (7-4) + (56-12)x + (7\cdot 8^2 - 4\cdot 3^2)\frac{x^2}{2!} + \left(\frac{7\cdot 8^3 - 4\cdot 3^3}{3!}\right)x^3 + \cdots$

$= 3 + 44x + 412\frac{x^2}{2!} + \cdots$

$\underline{3, 44, 412, 3476 \cdots}$ OR Sequence is $\underline{7\cdot 8^n - 4\cdot 3^n \quad n = 0, 1, 2, 3 \cdots}$

(e) $f(x) = 2e^{x} + 3x^2$

$= 2\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right] + 3x^2 = 2\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right] + 3\cdot 2!\,\frac{x^2}{2!} + \cdots$

$= 2 + 2x + 2\frac{x^2}{2!} + 2\frac{x^3}{3!} + \cdots + 3\cdot 2!\,\frac{x^2}{2!}$. sequence is $1, 2, (2+6), 2, 2, \cdots$

ie $1, 2, 6, 2, 2, \cdots$

(g) $f(x) = e^{3x} - 28x^3 - 6x^2 + 9x$.

$= 1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} + \cdots - 28x^3 - 6x^2 + 9x$

$= 1 + 3x + 3^2\cdot\frac{x^2}{2!} + 3^3\frac{x^3}{3!} + 3^4\frac{x^4}{4!} + \cdots - 28\cdot 3!\frac{x^3}{3!} - 6\cdot 2!\frac{x^2}{2!} + 9x$

The sequence is

$1, (3+9), (3^2 - 12), (3^3 - 28\times 3!), 3^4, 3^5, 3^6, \cdots$

$= 1, 12, 3, 141, 3^4, 3^5, 3^6$

$= 1, 12, -3, -141, 3^4, 3^5, 3^6$

(b)

(d)

(f) $f(x) = e^{x} + x^2$

$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots + x^2$

$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots + \frac{x^2}{2!}\cdot 2!$

$= 1 + x + (1+2)\frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ sequence is $1, 1, 3, 1, 1, \cdots$

Q) In each of the following, the function $f(x)$ is the generating function for the sequence $a_0, a_1, a_2 \ldots$ where as the sequence $b_0, b_1, b_2, \ldots$ is generated by the function $g(x)$. Express $g(x)$ in terms of $f(x)$

a) $b_3 = 3$
$b_n = a_n \quad n \in N \quad n \neq 3$

b) $b_3 = 3, \quad b_7 = 7$
$b_n = a_n, \quad n \in N \quad n \neq 3, 7$

c) $b_1 = 1$
$b_3 = 3$
$b_n = 2a_n \quad n \in N$
$n \neq 1, 3$

Ans:- (a) $f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$

$g(x) = \sum_{n=0}^{\infty} b_n x^n = b_0 x^0 + b_1 x^1 + b_2 x^2 + b_3 x^3 + b_4 x^4 + \ldots$

$= a_0 x^0 + a_1 x^1 + a_2 x^2 + 3x^3 + a_4 x^4 + \ldots$

$= f(x) - a_3 x^3 + 3x^3$

$= f(x) + (3 - a_3) x^3$

(b)

$f(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + \ldots$

$g(x) = b_0 x^0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + \ldots$

$= a_0 x^0 + a_1 x + a_2 x^2 + 3x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + 7x^7 + \ldots$

$= f(x) - a_3 x^3 - a_7 x^7 + 3x^3 + 7x^7$

$= f(x) + (3 - a_3) x^3 + (7 - a_7) x^7$

(c)

$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$

$g(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots + b_n x^n + \ldots$

$= 2a_0 + 1x + 2a_2 x^2 + 3x^3 \ldots + 2a_n x^n + \ldots$

$= 2f(x) - 2a_1 x - 2a_3 x^3 + x + 3x^3$

$= 2f(x) + (1 - 2a_1) x + (3 - 2a_3) x^3$

Q. In each of the following, the function $f(x)$ is the exponential generating function for the sequence $a_0, a_1, a_2 \ldots$ whereas the function $g(x)$ is the exponential generating function for the sequence $b_0, b_1, b_2 \ldots$. Express $g(x)$ in terms of $f(x)$

(a) $b_3 = 3$
$b_n = a_n \quad n \in N \quad n \neq 3$

(b) $a_n = 5^n$
$b_3 = -1$
$b_n = a_n \quad n \in N \quad n \neq 3$

H-W (c) $b_1 = 2, \; b_2 = 4, \quad b_n = 2a_n \quad n \in N, n \neq 1, 2$

Ans (a).
$$f(x) = a_0 + a_1 \frac{x}{1!} + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + a_4 \frac{x^4}{4!} + \cdots$$
$$g(x) = b_0 + b_1 x + b_2 \frac{x^2}{2!} + b_3 \frac{x^3}{3!} + b_4 \frac{x^4}{4!} + \cdots$$
$$= a_0 + a_1 x + a_2 \frac{x^2}{2!} + 3 \frac{x^3}{3!} + a_4 \frac{x^4}{4!} + \cdots$$
$$= f(x) - a_3 \cdot \frac{x^3}{3!} + 3 \frac{x^3}{3!} = \underline{\underline{f(x) + (3 - a_3) \frac{x^3}{3!}}}$$

(b)
$$f(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + a_4 \frac{x^4}{4!} + \cdots = 1 + 5x + \frac{(5x)^2}{2!} + \frac{(5x)^3}{3!} + \cdots = \underline{\underline{e^{5x}}}$$

$\boxed{f(x) = e^{5x}}$

$$g(x) = b_0 + b_1 x + b_2 \frac{x^2}{2!} + b_3 \frac{x^3}{3!} + b_4 \frac{x^4}{4!} + \cdots$$
$$= a_0 + a_1 x + a_2 \frac{x^2}{2!} - \frac{x^3}{3!} + a_4 \frac{x^4}{4!} + \cdots$$
$$= f(x) - a_3 \frac{x^3}{3!} - \frac{x^3}{3!} = f(x) - (5^3 + 1) \frac{x^3}{3!}$$
$$= \underline{\underline{e^{5x} - \left(\frac{126}{3!}\right) x^3}}$$

H-W (c)

Q) Find the exponential generating function for the sequence $0!, 1!, 2!, 3!, 4!, \dots$

$$f(x) = 0! + 1!\frac{x}{1!} + 2!\frac{x^2}{2!} + 3!\frac{x^3}{3!} + \dots$$

$$= 1 + x + x^2 + x^3 + \dots \quad = (1-x)^{-1} = \frac{1}{1-x}$$

Q) Find the sequence generated by each of the following exponential generating functions.

(a) $f(x) = \frac{1}{1-x}$ (b) $f(x) = e^{2x} - 3x^3 + 5x^2 + 7x$

(c) $f(x) = 3/(1-2x) + e^x$

Ans:- (a) $f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$

$$= 1 + x + 2!\frac{x^2}{2!} + 3!\frac{x^3}{3!} + \dots$$

sequence is $0!, 1!, 2!, 3!, \dots$

(c) $f(x) = \frac{3}{1-2x} + e^x = 3\left(1 + 2x + (2x)^2 + (2x)^3 + \dots\right) + \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots\right)$

$$= \left(3 + 6x + 3\cdot 2^2\cdot 2!\frac{x^2}{2!} + 3\cdot 2^3\cdot 3!\cdot\frac{x^3}{3!} + \dots\right) + \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)$$

$$= 4 + 7x + \left((3\cdot 2^2\cdot 2) + 1\right)\frac{x^2}{2!} + \left[(3\cdot 2^3\cdot 3!) + 1\right]\frac{x^3}{3!} + \dots$$

$$= 4 + 7x + 25\frac{x^2}{2!} + 145\frac{x^3}{3!} + \dots$$

$\therefore$ sequence is $4, 7, 25, 145, \dots$

(b) $f(x) = e^{2x} - 3x^3 + 5x^2 + 7x = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \dots - 3x^3 + 5x^2 + 7x$

$$= 1 + 2x + 2^2\cdot\frac{x^2}{2!} + 2^3\cdot\frac{x^3}{3!} + \dots - 3\cdot 3!\,\frac{x^3}{3!} + 5\cdot 2!\,\frac{x^2}{2!} + 7x$$

$= 1 + 9x + 14\frac{x^2}{2!} - 10\frac{x^3}{3!} + 2^4\cdot\frac{x^4}{4!} + \dots$ $\therefore$ sequence is $1, 9, 14, -10, 2^4, 2^5, \dots$

III RECURRENCE RELATIONS OR DIFFERENCE EQUATIONS

For a sequence $a_0, a_1, a_2, \cdots a_n, \cdots$ an eqn relating a_n for any n to one or more of the a_i's $i<n$ is called recurrence relation or difference equation.

Eg:- Consider the sequence $5, 15, 45, 135, \cdots$

ie, $a_n = 5 \cdot 3^n$ for $n \geq 0$

$a_0 = 5 \cdot 3^0 = 5$
$a_1 = 5 \cdot 3^1 = 15$
$a_2 = 5 \cdot 3^2 = 45$
$\vdots$

Also $\boxed{a_{n+1} = 3a_n \quad n \geq 0 \quad \text{with } a_0 = 5}$

Completely satisfies the ~~numeric~~ sequence $5, 15, 45, 135, \cdots$

This relation is the recurrence relation for the sequence $5, 15, 45, 135, \cdots$ OR $\boxed{a_n = 3a_{n-1} \ n \geq 1, \text{ with } a_0 = 5}$ is also correct

Note:-

* Since a_{n+1} depends only on its immediate predecessor the relation is said to be first order.
* The given values of a sequence (such as $a_0 = 5$) are known as boundary conditions.
* $a_n = 5 \cdot 3^n$ for $n \geq 0$ is called the unique solution of the given recurrence relation.

Eg:- Write the recurrence relation for the fibonacci series $1, 1, 2, 3, 5, 8, 13, 21, 44, \cdots$

$$a_n = a_{n-1} + a_{n-2} \quad n \geq 2 \qquad \text{OR} \qquad a_{n+2} = a_{n+1} + a_n \quad n \geq 0$$

with $a_0 = 1$
$a_1 = 1$

III] RECURRENCE RELATIONS OR DIFFERENCE EQUATIONS

For a sequence $a_0, a_1, a_2, \dots a_n, \dots$ an eqn relating a_n for any n to one or more of the a_i's $i<n$ is called recurrence relation or difference equation.

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Also $\boxed{a_{n+1} = 3a_n \quad n \geq 0 \quad \text{with } a_0 = 5}$

Completely satisfies the ~~numeric~~ sequence $5, 15, 45, 135, \dots$

This relation is the recurrence relation for the sequence $5, 15, 45, 135, \dots$ OR $\boxed{a_n = 3a_{n-1} \quad n \geq 1, \quad \text{with } a_0 = 5}$ is also correct

Note:-

* Since a_{n+1} depends only on its immediate predecessor the relation is said to be first order.
* The given values of a sequence (such as $a_0 = 5$) are known as boundary conditions.
* $a_n = 5\cdot 3^n$ for $n \geq 0$ is called the unique solution of the given recurrence relation.

Eg:- Write the recurrence relation for the fibonacci series $0, 1, 1, 2, 3, 5, 8, 13, 21, 44, \dots$ (Order 2)

$$a_n = a_{n-1} + a_{n-2} \quad n \geq 2 \qquad \text{OR} \qquad a_{n+2} = a_{n+1} + a_n \quad n \geq 0$$

with $a_0 = 0$
$a_1 = 1$

RESULT (Geometric progression)

The recurrence relation $\boxed{a_{n+1} = d\,a_n}$ → Recurrence relation. $n \geq 0$, $a_0 = A$, d is a constant

has a unique solution $a_n = A d^n$ $n \geq 0$

OR $\boxed{a_n = a_0 d^n}$ → solution

1) Solve the recurrence relation $a_n = 7a_{n-1}$ $n \geq 1$ and $a_2 = 98$

Ans:- This is same as $a_{n+1} = 7a_n$ $n \geq 0$, $a_2 = 98$. where $d = 7$

$\therefore$ Solution is $a_n = a_0 d^n$

$a_n = a_0 7^n$ (We are given a_2. So find a_0, using a_2)

ie $a_2 = a_0 \cdot 7^2$

$98 = a_0 49$.

$\therefore a_0 = \frac{98}{49} = 2$

$\therefore$ unique solution of the given recurrence relation is $a_n = 2 \cdot 7^n$ $n \geq 0$

2) Find a unique solution of the recurrence relation

$6a_n - 7a_{n-1} = 0$ $n \geq 1$, $a_3 = 343$

Ans:- $a_n = \frac{7}{6} a_{n-1}$

Solution is $a_n = a_0 \left(\frac{7}{6}\right)^n$. find a_0 using a_3.

$a_3 = a_0 \cdot \left(\frac{7}{6}\right)^3$ ie $343 = a_0 \cdot \left(\frac{7}{6}\right)^3 \Rightarrow a_0 = 216$

$\therefore$ unique solution is $a_n = 216 \times \left(\frac{7}{6}\right)^n$ $n \geq 0$

3) Find a recurrence relation with initial condition that uniquely determines each of the following sequences

ⓐ 3, 7, 11, 15, 19, ...
(increases by 4)

ⓑ 8, $\frac{24}{7}$, $\frac{72}{49}$, $\frac{216}{343}$, ...

Ans:- ⓐ $a_n = a_{n-1} + 4$ $n \geq 1$ with $a_0 = 3$ is the recurrence relation

ⓑ $a_n = \frac{3}{7} a_{n-1}$ $n \geq 1$ with $a_0 = 8$

4) If a_n, $n \geq 0$ is the unique solution of the recurrence relation $a_{n+1} - d a_n = 0$ and $a_3 = \frac{153}{49}$, $a_5 = \frac{1377}{2401}$. What is d.

Ans:- unique solution $a_n = a_0 d^n$

We have $a_3 = a_0 d^3 \Rightarrow \frac{153}{49} = a_0 d^3$ —①

$a_5 = a_0 d^5 \Rightarrow \frac{1377}{2401} = a_0 d^5$ —②

$$\frac{①}{②} \Rightarrow \frac{\cancel{153}}{\cancel{49}} \times \frac{\overset{49}{\cancel{2401}}}{\underset{9}{\cancel{1377}}} = \frac{a_0 d^3}{a_0 d^5}$$

$$\frac{49}{9} = \frac{1}{d^2} \qquad d^2 = \frac{9}{49}$$

$$\therefore d = \pm\frac{3}{7}$$

5) H.W Find a unique solution for each of the following recurrence relations.

(a) $a_{n+1} - (1.5)a_n = 0 \quad n \geq 0$

(b) $2a_n - 3a_{n-1} = 0 \quad n \geq 1, \quad a_4 = 81$

Ans: (a) It is a geometric progression with $d = 1.5$

$\therefore$ unique solution $a_n = a_0(1.5)^n \geq 0$.

(no bdry conditions are given $\therefore$ not able to find a_0)

(b) $2a_n - 3a_{n-1} = 0, \quad a_4 = 81$

$a_n - \frac{3}{2}a_{n-1} = 0$

$\therefore$ unique solution $a_n = a_0 \cdot \left(\frac{3}{2}\right)^n$ find a_0 using $a_4 = 81$.

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IV SECOND ORDER LINEAR HOMOGENEOUS RECURRENCE RELATION WITH CONSTANT COEFFICIENTS

1) **Defn:- Linear recurrence relation of order k.**

$$\underset{\neq 0}{C_0} a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + \underset{\neq 0}{C_k} a_{n-k} = f(n)$$

where $n \geq 0$

$n \geq k$

$k \in Z^+$

$(C_0 \neq 0), C_1, C_2 \cdots (C_k \neq 0)$ are real numbers

2) **Homogeneous & Non homogeneous linear recurrence relations**

When RHS, $f(n) \neq 0$, relation is non homogeneous

When RHS, $f(n) = 0$, relation is homogeneous.

Note:- In this section we concentrate on HOMOGENEOUS relation of ORDER TWO

$$\boxed{C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} = 0 \qquad n \geq 2}$$

It is of order 2, because a_n depends on the previous 2 terms, a_{n-1} & a_{n-2}.

$\boxed{RHS=0}$

Method for finding homogeneous 2nd order linear recurrence relation

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} = 0$$

Step 1: Write the characteristic equation by substituting $a_n = r^2$, (because it is of order 2)
$a_{n-1} = r^1$
$a_{n-2} = r^0$

ie $C_0 r^2 + C_1 r + C_2 = 0$ is the characteristic equation

It is a quadratic equation and
r_1 and r_2 are called characteristic roots

Case (A) (r_1 and r_2 are distinct & real)

general solution $\boxed{a_n = A_1 r_1^n + A_2 r_2^n}$ A_1, A_2 are arbitrary constants.

Case (B) (r_1 and r_2 are real and repeated
ie $r_1 = r_2 = r$)

general solution $\boxed{a_n = (A_1 + A_2 n) r^n}$ A_1, A_2 are arbitrary constants.

Case (C) (r_1 and r_2 are complex roots
ie $\alpha \pm i\beta$)
$r_1 = \alpha + i\beta$
$r_2 = \alpha - i\beta$

general solution $\boxed{a_n = A_1 (\alpha + i\beta)^n + A_2 (\alpha - i\beta)^n}$ A_1, A_2 Arb. const.

1. Solve the recurrence relation

$a_n + a_{n-1} - 6a_{n-2} = 0 \quad n \geq 2, \quad a_0 = -1, \quad a_1 = 8$

Ans:- Ch. eqn is $r^2 + r - 6 = 0$

$(r+3)(r-2) = 0$

$r = -3, 2.$

order 2

$a_n = r^2$

$a_{n-1} = r^1$

$a_{n-2} = r^0$

$\therefore$ general solution $a_n = A_1(-3)^n + A_2 2^n \longrightarrow ①$

find A_1 & A_2

The arb. constants

$a_0 = -1$ (given)

$① \Rightarrow a_0 = A_1(-3)^0 + A_2(2)^0$

$-1 = A_1 + A_2 \rightarrow ②$

$a_1 = 8$ (given)

$① \Rightarrow a_1 = A_1(-3)^1 + A_2(2)^1$

$8 = -3A_1 + 2A_2 \rightarrow ③$

Solve ② & ③

$A_2 = 1, \; A_1 = -2$

$\therefore$ general solution is ~~$a_n = (-3)^n - 2(2)^n$~~

$a_n = -2(-3)^n + 2^n$

$a_n = 2^n - 2(-3)^n \quad n \geq 0$

is the unique solution.

HW

2. Solve the recurrence relation for Fibonacci sequence.

Ans:- 0, 1, 1, 2, 3, 5, 8, 13, ... is the Fibonacci sequence

Recurrence relation is $a_n = a_{n-1} + a_{n-2} \quad n \geq 2$

$a_0 = 0$

$a_1 = 1$

ie $a_n - a_{n-1} - a_{n-2} = 0$ with $a_0 = 0$ $a_1 = 1$

Ch eqn is $r^2 - r - 1 = 0$

$r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$ ie $\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$

gen soln is $a_n = A_1\left(\frac{1+\sqrt5}{2}\right)^n + A_2\left(\frac{1-\sqrt5}{2}\right)^n$ —— (1)

given $a_0 = 0$

$a_0 = A_1 + A_2$

$0 = A_1 + A_2$ —— (2)

$a_1 = 1$

$a_1 = A_1\left(\frac{1+\sqrt5}{2}\right) + A_2\left(\frac{1-\sqrt5}{2}\right)$

$1 = A_1\left(\frac{1+\sqrt5}{2}\right) + A_2\left(\frac{1-\sqrt5}{2}\right)$ —— (3)

from (2) $\Rightarrow A_1 = -A_2$

put it in (3)

$1 = -A_2\left(\frac{1+\sqrt5}{2}\right) + A_2\left(\frac{1-\sqrt5}{2}\right)$

$1 = A_2\left(\frac{-1-\sqrt5+1-\sqrt5}{2}\right)$

$1 = A_2\left(-\frac{2\sqrt5}{2}\right)$

ie $1 = A_2(-\sqrt5)$ $\therefore A_2 = \frac{-1}{\sqrt5}$

$A_1 = -A_2$

$\therefore A_1 = \frac{1}{\sqrt5}$

$\therefore$ gen solution is $a_n = \frac{1}{\sqrt5}\left(\frac{1+\sqrt5}{2}\right)^n - \frac{1}{\sqrt5}\left(\frac{1-\sqrt5}{2}\right)^n$

Note:-

Solve the recurrence relation

$$F_{n+2} = F_{n+1} + F_n \quad n \geq 0, \quad F_0 = 0, F_1 = 1$$

~~Ch eqn is~~ ie $F_{n+2} - F_{n+1} - F_n = 0$

$F_{n+2} = \gamma^2$

$F_{n+1} = \gamma$

$F_n = \gamma^0$

Ch. eqn is $\gamma^2 - \gamma - 1 = 0$

is same as the Fibonacci recurrence relation.

$\therefore$ Ans is same as above

3. Solve the recurrence relation

$$a_{n+2} = 4a_{n+1} - 4a_n \quad n \geq 0 \quad a_0 = 1,\ a_1 = 3$$

Ans:- Second order difference eqn.

$$a_{n+2} - 4a_{n+1} + 4a_n = 0$$

$a_{n+2} = r^2$, $a_{n+1} = r^1$, $a_n = r^0$

$$r^2 - 4r + 4 = 0$$

$$(r-2)^2 = 0$$

$r = 2, 2$ (a root of multiplicity 2) ie $r = 2$ is repeated twice.

$\therefore$ gen soln $a_n = (A_1 + A_2 n) 2^n$ —— ①

given $a_0 = 1$

$$a_0 = (A_1 + A_2 \times 0) 2^0$$

$$\underline{\underline{1 = A_1}}$$

$a_1 = 3$

$$a_1 = (A_1 + A_2) 2$$

$$3 = (1 + A_2) 2$$

$$\frac{3}{2} = 1 + A_2$$

$$A_2 = \frac{3}{2} - 1 = \frac{1}{2}$$

$\therefore$ ① $\rightarrow$ gen soln is $\underline{\underline{\left(1 + \frac{1}{2}n\right) 2^n = a_n}}$

4. Solve the recurrence relation

$$2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n \quad n \geq 0, \quad a_0 = 0,\ a_1 = 1,\ a_2 = 2$$

Ans:- 3^{rd} order recurrence relation.

$a_{n+3} = r^3$, $a_{n+2} = r^2$, $a_{n+1} = r$, $a_n = r^0$

$$2a_{n+3} - a_{n+2} - 2a_{n+1} + a_n = 0$$

Ch. eqn is $2r^3 - r^2 - 2r + 1 = 0$

$$r = \frac{1}{2}, 1, -1$$

gen. soln is $a_n = A_1\left(\frac{1}{2}\right)^n + A_2(1)^n + A_3(-1)^n$ —— (1)

given $a_0 = 0$ $a_1 = 1$ $a_2 = 2$

$0 = A_1 + A_2 + A_3 \rightarrow (2)$

$1 = \frac{1}{2}A_1 + A_2 - A_3 \rightarrow (3)$

$2 = A_1 \cdot \frac{1}{4} + A_2 + A_3 \rightarrow (4)$

Solving (2), (3), (4)

$A_1 = \frac{-8}{3}$

$A_2 = \frac{5}{2}$

$A_3 = \frac{1}{6}$

$(2)+(3) \Rightarrow \frac{3}{2}A_1 + 2A_2 = 1$ —— (5)

$(3)+(4) \Rightarrow \frac{3}{4}A_1 + 2A_2 = 3$ —— (6)

subtraction $(5)-(6) \Rightarrow$ $\frac{3}{4}A_1 = -2$

$\therefore A_1 = \frac{-8}{3}$

from (5) $A_2 = \frac{5}{2}$

put in (2) $A_3 = \frac{1}{6}$

$\therefore$ gen soln is $a_n = \underline{\underline{\left(\frac{-8}{3}\right)\left(\frac{1}{2}\right)^n + \frac{5}{2} + \frac{1}{6}(-1)^n}}$ $n \geq 0$

V.Imp When the roots are complex

* $e^{i\theta} = \cos\theta + i\sin\theta$

DE-MOIVRE'S THEOREM

$(e^{i\theta})^n = e^{in\theta} = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

* If $z = x + iy$ we can write

$\boxed{z = r(\cos\theta + i\sin\theta)}$ where $r = \sqrt{x^2+y^2}$ & $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\tan\theta = \frac{y}{x}$ when $x \neq 0$

* If $z = iy$ $(x=0)$, $y > 0$

$\boxed{z = y\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)}$

* If $z = yi$ $(x=0)$, $y < 0$

$\boxed{z = |y|\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)}$

In all cases $z^n = r^n(\cos n\theta + i\sin n\theta)$

1) Solve the recurrence relation

$$a_n = 2(a_{n-1} - a_{n-2}) \text{ where } n \geq 2, \quad a_0 = 1, \; a_1 = 2$$

Ans:- $a_n - 2a_{n-1} + 2a_{n-2} = 0$

Ch. eqn is $r^2 - 2r + 2 = 0$

$$r = 1 \pm i$$

$\therefore$ gen. soln $a_n = A_1(1+i)^n + A_2(1-i)^n$

$x + iy = r(\cos\theta + i\sin\theta)$ $\quad r = \sqrt{x^2+y^2}$ $\quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ $\quad r = \sqrt{1+1} = \sqrt{2}$ $\quad \theta = \tan^{-1}\frac{1}{1}$

$1-i = \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$ $\quad = \tan^{-1}1 = \frac{\pi}{4}$

$$\therefore a_n = A_1\left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^n + A_2\left[\sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)\right]^n$$

$$= A_1(\sqrt{2})^n\left[\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right] + A_2(\sqrt{2})^n\left[\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right]$$

$$= (\sqrt{2})^n(A_1+A_2)\cos\frac{n\pi}{4} + (\sqrt{2})^n(A_1-A_2)i\sin\frac{n\pi}{4}$$

$$= (\sqrt{2})^n\left[(B_1)\cos\frac{n\pi}{4} + B_2\sin\frac{n\pi}{4}\right] \longrightarrow (1)$$

$B_1 = A_1 + A_2$

$B_2 = (A_1 - A_2)i$

$a_0 = 1$, $\quad 1 = B_1\cos 0 + B_2\cdot\sin 0$ $\quad \therefore B_1 = 1$

$a_1 = 2$, $\quad 2 = (\sqrt{2})\left[B_1\cos\frac{\pi}{4} + B_2\sin\frac{\pi}{4}\right] \Rightarrow 2 = \sqrt{2}\left[\frac{1}{\sqrt{2}} + B_2\frac{1}{\sqrt{2}}\right]$

$2 = 1 + B_2$ $\quad \therefore B_2 = 1$

(1) $\Rightarrow$ gen soln $a_n = (\sqrt{2})^n\left[\cos\frac{n\pi}{4} + \sin\frac{n\pi}{4}\right]$ $\quad n \geq 0$

2) Solve $a_{n+2}+a_n=0 \quad n\geq 0, \quad a_0=0, \ a_1=3$

Ch. eqn is: $r^2+1=0$

$r^2=-1$

$r=\pm i$

$\therefore$ gen solution is $a_n = A_1(i)^n + A_2(-i)^n$

$$\boxed{z=iy \quad y>0 \quad z=y\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)}$$

Consider $z=i$ $\quad y=1, \quad z=1\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$

$= i\sin\frac{\pi}{2}$

S | A
T | C

Consider $z=-i$ $\quad y=-1, \quad z=(+1)\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)$

$= i\sin 3\frac{\pi}{2} = -i\sin\frac{\pi}{2}$

$a_n = A_1\left(i\sin\frac{\pi}{2}\right)^n + A_2\left(-i\sin\frac{\pi}{2}\right)^n$

$= A_1\left(i\sin\frac{n\pi}{2}\right) + A_2\left(-i\sin n\frac{\pi}{2}\right)$

$= i(A_1-A_2)\sin n\frac{\pi}{2} = B\sin n\frac{\pi}{2} \rightarrow (1) \quad B=(A_1-A_2)i$

$a_0=0$ as given

$0 = B\cdot\sin 0$

$a_1=3$

$3 = B\sin\frac{\pi}{2}$

$B=3$

$\therefore$ gen solution $a_n = 3\sin n\frac{\pi}{2} \quad n\geq 0$

IV THE NONHOMOGENEOUS RECURRENCE RELATION

We now consider the nonhomogeneous first and 2nd order recurrence relation.

ie $a_n + c_1 a_{n-1} = f(n) \quad n \geq 1$ when $RHS = f(n) \neq 0$

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} = f(n) \quad n \geq 2$$

$$\boxed{\text{The general solution } a_n = a_n^{(h)} + a_n^{(p)}}$$

where $a_n^{(h)}$: gen. solution of homogeneous relation

$a_n^{(p)}$: a particular solution.

Particular Solutions.

Case ① When $\boxed{f(n) = RHS = k\beta^n}$ k:- constant

$a_n^{(p)} = B.\beta^n$ when β is not a solution of ~~hom. relation~~ ch. eqn.

$a_n^{(p)} = B.n^m\beta^n$ when β is a root of the ch. eqn with multiplicity m.

where B is a constant to be determined

1. Solve $a_n - 3a_{n-1} = 5(7^n)$ $n \geq 1$ and $a_0 = 2$

Ans- $a_n = a_n^{(h)} + a_n^{(p)}$

To find $a_n^{(h)}$ ch. eqn is $r - 3 = 0$

ch. root is $r = 3$.

$$\therefore a_n^{(h)} = A3^n$$

To find $a_n^{(p)}$ RHS $= 5 \cdot 7^n$ (7 is not a ch. root)

$$\therefore a_n^{(p)} = B \cdot 7^n$$ (find the constant B)

$a_n^{(p)}$ is a particular solution.

So substitute in the given eqn.

put $a_n = B7^n$

$a_{n-1} = B \cdot 7^{n-1}$

ie $a_n - 3a_{n-1} = 5(7^n)$

$$B \cdot 7^n - 3 \cdot B7^{n-1} = 5 \times 7^n$$ (Cancel 7^{n-1} from all terms)

$$B7 - 3B = 5 \times 7$$

$$4B = 35 \qquad \therefore B = \frac{35}{4}$$

$$\therefore a_n^{(p)} = \frac{35}{4} 7^n$$

$\therefore$ gen solution $a_n = a_n^{(h)} + a_n^{(p)}$

$$a_n = A \times 3^n + \frac{35}{4} 7^n$$ with $a_0 = 2$ (using a_0 find A)

ie $a_0 = A3^0 + \frac{35}{4} \cdot 7^0$

$$2 = A + \frac{35}{4} \Rightarrow 8 = 4A + 35 \Rightarrow 4A = 8 - 35 = -27$$

$$\therefore A = \frac{-27}{4}$$

$$\therefore a_n = \frac{-27}{4} 3^n + \frac{35}{4} 7^n \qquad n \geq 0$$

Very important

2. Solve $a_n - 3a_{n-1} = 5(3^n)$ $n \geq 1$ $a_0 = 2$

Ans:- $a_n = a_n^{(h)} + a_n^{(p)}$

$\underline{a_n^{(h)}}$ Ch. eqn is $r - 3 = 0$

Ch. root is $r = 3$

$\therefore a_n^{(h)} = A \cdot 3^n$

$\underline{a_n^{(p)}}$ $f(n) = 5(3^n)$ 3 is a ch. root with multiplicity 1.

$\therefore a_n^{(p)} = B \cdot n \cdot 3^n$ (find B)

Substitute in the given recurrence relation.

$B \cdot n \cdot 3^n - 3 \cdot B(n-1)3^{n-1} = 5 \times 3^n$

put $a_n = B n 3^n$
$a_{n-1} = B(n-1)3^{n-1}$

$B \cdot n 3 - 3B(n-1) = 5 \times 3$

$3Bn - 3Bn + 3B = 15$

$3B = 15$ $\therefore B = 5$

$\therefore a_n^{(p)} = 5n \cdot 3^n$

$\underline{a_n} = A 3^n + 5n 3^n$ with $a_0 = 2$ (find A using a_0)

$a_0 = A \times 3^0 + 5 \times 0 \times 3^0$

$2 = A.$

$\therefore a_n = 2 \cdot 3^n + 5n 3^n$

$\therefore a_n = (2 + 5n)3^n, \quad n \geq 0$ → very important

3. $a_{n+1} = 2a_n + 2 \quad n \geq 1$ with $a_1 = 1$

Ans:- $a_{n+1} - 2a_n = 2.$

$a_n^{(h)}$ ch. eqn is $r - 2 = 0$

$r = 2.$

$\therefore a_n^{(h)} = A \cdot 2^n$

$a_n^{(p)}$ RHS $= 2 = 2 \times 1^n$ 1 is not a ch. root.

$a_n^{(p)} = B \cdot 1^n$ find B.

$a_{n+1} - 2a_n = 2.$

$B - 2B = 2$

$-B = 2 \quad \therefore B = -2.$

$a_n^{(p)} = -2.$

$a_n = A \cdot 2^n - 2.$ with $a_1 = 1$

$a_1 = A \cdot 2^1 - 2$

$1 = 2A - 2.$

$2A = 1 + 2 \quad \therefore A = \frac{3}{2}$

$\therefore a_n = \frac{3}{2} \times 2^n - 2 \quad n \geq 0$

H.W

4. Solve $a_n = 2a_{n-1} + 4^{n-1} \quad n \geq 2, \; a_1 = 3$

Ans:- $a_n - 2a_{n-1} = 4^{n-1}$

$a_n - 2a_{n-1} = \frac{1}{4} \cdot 4^n$

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5. Solve $a_{n+2} - 8a_{n+1} + 16a_n = 8(5^n) + 6(4^n) \quad n \geq 0 \quad \longrightarrow (1)$

$a_0 = 12$

$a_1 = 5$

Ans: $\underline{a_n^{(h)}}$

Ch. eqn is $r^2 - 8r + 16 = 0$

$(r-4)^2 = 0 \quad r = 4, 4$ are ch. roots.

$$a_n^{(h)} = (A_1 + A_2 n) 4^n$$

$\underline{a_n^{(p)}}$

$$a_n^{(p)} = B_1 5^n + B_2 \cdot n^2 \cdot 4^n$$

(n^2 is because, $r = 4$ is a ch. root with multiplicity 2)

To find B_1, B_2 (substitute in (1))

$$B_1 5^{n+2} + B_2 (n+2)^2 4^{n+2} - 8\left[B_1 5^{n+1} + B_2 (n+1)^2 4^{n+1}\right] + 16\left[B_1 5^n + B_2 n^2 4^n\right] = 8 \cdot 5^n + 6 \cdot 4^n$$

$$\left[B_1 \cdot 5^{n+2} - 8B_1 5^{n+1} + 16B_1 5^n\right] + \left[B_2 (n+2)^2 4^{n+2} - 8B_2 (n+1)^2 4^{n+1} + 16B_2 n^2 4^n\right] = 8 \cdot 5^n + 6 \cdot 4^n$$

Comparing co-efficients of 5^n & 4^n on both sides.

$B_1 \cdot 5^2 - 8B_1 \cdot 5 + 16B_1 = 8$

$25B_1 - 40B_1 + 16B_1 = 8$

$41B_1 - 40B_1 = 8$

$\underline{B_1 = 8}$

& $B_2 (n+2)^2 4^2 - 8B_2 (n+1)^2 4 + 16B_2 n^2 = 6$

$B_2 \left(16(n^2 + 4n + 4) - 32(n^2 + 2n + 1) + 16n^2\right) = 6$

$B_2 (32n^2 - 32n^2 + 64n - 64n + 64 - 32) = 6$

$B_2 (32) = 6 \quad \therefore B_2 = \frac{6}{32} = \frac{3}{16}$

$$\therefore a_n^{(p)} = 8 \cdot 5^n + \frac{3}{16} n^2 \cdot 4^n$$

$$a_n = (A_1 + A_2 n) 4^n + 8 \cdot 5^n + \frac{3}{16} n^2 (4^n) \quad n \geq 0 \quad a_0 = 12, a_1 = 5$$

$a_0 = A_1 + 8$

$12 = A_1 + 8$

$\underline{A_1 = 4}$

$a_1 = (4 + A_2) 4 + 40 + \frac{3}{4}$

$5 = 16 + 4A_2 + 40 + \frac{3}{4}$

$4A_2 = -207$

$A_2 = -207/4$

Gn soln: $a_n = \left(4 - \frac{207}{4} n\right) 4^n + 8(5^n) + \frac{3}{16} n^2 (4^n) \quad n \geq 0$

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5. Solve $a_{n+2} = 4a_{n+1} - 3a_n - 200$, $n \geq 0$, $a_0 = 3000$, $a_1 = 3300$

Ans:- $a_{n+2} - 4a_{n+1} + 3a_n = -200$ —— ①

$a_n^{(h)}$ Ch. eqn is $r^2 - 4r + 3 = 0$

$(r-3)(r-1) = 0$

ch. root $r = 3, 1$

$\therefore a_n^{(h)} = A_1 3^n + A_2 1^n$

$a_n^{(p)}$ RHS = $-200 = -200 \cdot 1^n$

$\therefore a_n^{(p)} = B.n.1^n$ ($\because$ 1 is a ch. root of multiplicity 1)

Substitute in —— ①

$B(n+2) - 4B(n+1) + 3Bn = -200$

$\cancel{Bn} + 2B - \cancel{4Bn} - 4B + \cancel{3Bn} = -200$

$-2B = 200 \quad \therefore B = 100$

$\therefore a_n^{(p)} = 100n$

$a_n = a_n^{(h)} + a_n^{(p)}$

$a_n = A_1 3^n + A_2 + 100n \qquad a_0 = 3000, \; a_1 = 3300$

$a_0 = A_1 + A_2 \Rightarrow A_1 + A_2 = 3000$ —— ②

$a_1 = 3A_1 + A_2 + 100$

$3300 = 3A_1 + A_2 + 100 \Rightarrow 3A_1 + A_2 = 3200$ —— ③

Solving ② & ③ $\quad 2A_1 = 200 \quad A_1 = 200/2 = 100$

$A_2 = 2900$

$\therefore a_n = 100\,3^n + 2900 + 100n \qquad n \geq 0$

CASE II WHEN $RHS = f(n) = n^t . \beta^n$

ie $RHS = f(n) = \begin{pmatrix} \text{polynomial} \\ \text{of degree } t \end{pmatrix} \times \beta^n$ form.

* particular solution $a_n^{(P)} = n^m \times (B_t n^t + B_{t-1} n^{t-1} + \dots + B_1 n + B_0) . \beta^n$

if β is a ch. root of of multiplicity m.

* Particular solution $a_n^{(P)} = (B_t n^t + B_{t-1} n^{t-1} + \dots + B_1 n + B_0) \beta^n$

if β is not a ch. root of the ch. eqn.

(where $B_0, B_1, B_2, \dots$ are constants to be determined)

1. Solve $a_{n+1} - a_n = n$ $\quad n \geq 2, \ a_2 = 1$ —— (1)

Ans:- $a_n^{(h)}$:- ch. eqn is $r - 1 = 0$

ch. root is $r = 1$

$a_n^{(h)} = A \cdot 1^n = A$

$a_n^{(P)}$:- $RHS = n \cdot 1^n$ $\begin{pmatrix} \text{A polynomial} \\ \text{of degree 1} \end{pmatrix} \times 1^n$

1 is a ch. root of multiplicity 1

$\therefore a_n^{(P)} = n^1 (B_1 n + B_0)$

$a_n^{(P)} = B_1 n^2 + B_0 n$

Now find B_0 & B_1 →

put $a_n = B_1 n^2 + B_0 n$

$a_{n+1} = B_1 (n+1)^2 + B_0 (n+1)$

in eqn (1)

to find B_1 & B_0

$\therefore$ ① $\Longrightarrow$ $B_1(n+1)^n + B_0(n+1) - (B_1 n^2 + B_0 n) = n$

Equate the co-efficients of n^2, n and constant terms.

$$B_1(n^2+2n+1) + \cancel{B_0 n} + B_0 - \cancel{B_1 n^2} - \cancel{B_0 n} = n$$

<u>constants</u> $\quad B_1 + B_0 = 0$

<u>co-efficient of n</u> $\quad 2B_1 = 1 \quad \therefore B_1 = 1/2$.

$\therefore B_0 = -\frac{1}{2}$.

$$\therefore \quad a_n^{(P)} = \frac{1}{2}n^2 - \frac{1}{2}n = \frac{n}{2}(n-1)$$

$\therefore$ general soln $\quad a_n = a_n^{(h)} + a_n^{(P)}$

$= A + \frac{n}{2}(n-1) \qquad n \geq 2$

$a_2 = 1$

Now find arbitrary constant A

ie $\quad a_2 = A + \frac{2}{2}(2-1)$

$1 = A + 1 \times 1$

ie $\underline{A = 0}$

$\therefore$ general soln $\underline{a_n = \frac{n}{2}(n-1) \quad n \geq 2}$

2. Solve $a_{n+1} - a_n = 2n+3 \quad n \geq 0, \; a_0 = 1$ —— ①

Ans:- $\underline{a_n^{(h)}}$ ch. eqn, $r - 1 = 0$

ch. root, $r = 1$

$\therefore a_n^{(h)} = A \cdot 1^r = \underline{\underline{A}}$

$\underline{a_n^{(p)}}$ RHS $= (2n+3) \cdot 1^n$

$\downarrow$

$\begin{pmatrix} \text{A polynomial of} \\ \text{degree } 1 \end{pmatrix} \times \beta^n$ form

$\therefore a_n^{(p)} = n(B_1 n + B_0) \cdot 1^n$

(multiplication by n is because, multiplicity of ch. root $r = 1$ is 1)

find B_0 & B_1

put $\left.\begin{array}{l} a_n = B_1 n^2 + B_0 n \\ a_{n+1} = B_1 (n+1)^2 + B_0 (n+1) \end{array}\right\}$ in eqn ①

$\left(B_1 (n+1)^2 + B_0 (n+1)\right) - \left(B_1 n^2 + B_0 n\right) = 2n+3$

$B_1 (n^2 + 2n + 1) + \cancel{B_0 n} + B_0 - B_1 n^2 - \cancel{B_0 n} = 2n+3$

$\cancel{B_1 n^2} + 2B_1 n + B_1 + B_0 - \cancel{B_1 n^2} = 2n+3$

co-efft of n : $2B_1 = 2 \Rightarrow B_1 = 1$

constant : $\underline{B_1 + B_0 = 3} \Rightarrow B_0 = 2$

~~solving~~ $\therefore \underline{\underline{a_n^{(p)} = n(n+2)}}$

$\therefore$ general soln $a_n = a_n^{(h)} + a_n^{(p)}$

$a_n = A + n(n+2) \quad n \geq 0, \; a_0 = 1$

$\left.\begin{array}{l} \text{find the arb.} \\ \text{constant } A \end{array}\right\}$ $a_0 = A + 0(2)$

$1 = A \quad \therefore \underline{\underline{A = 1}}$

$\therefore a_n = 1 + n(n+2) \quad n \geq 0$

$= n^2 + 2n + 1$

$\underline{a_n = (n+1)^2, \quad n \geq 0}$

3. Find the general solution for the recurrence relation $a_{n+3} - 3a_{n+2} + 3a_{n+1} - a_n = 3 + 5n \quad n \geq 0$ → ①

Ans:- $a_n^{(h)}$

Ch. eqn is $r^3 - 3r^2 + 3r - 1 = 0$

$(r-1)^3 = 0$

Ch. roots, $r = 1, 1, 1$ ($r=1$ multiplicity 3)

$\therefore a_n^{(h)} = (A_0 + A_1 n + A_2 n^2)\, 1^n$

$a_n^{(h)} = A_0 + A_1 n + A_2 n^2$

$a_n^{(p)}$

$RHS = (3+5n) \times 1^n$

(A polynomial of degree 1) $\times \beta^n$

$\beta = 1$ is a ch. root with multiplicity 3.

$\therefore a_n^{(p)} = n^3 (B_1 n + B_0) \cdot 1^n$

To find B_0 & B_1 →

put $a_{n+1} = (n+1)^3 (B_1(n+1) + B_0)$

$a_{n+2} = (n+2)^3 (B_1(n+2) + B_0)$

$a_{n+3} = (n+3)^3 (B_1(n+3) + B_0)$

$a_n = n^3 (B_1 n + B_0)$

} in eqn ①

① ⟹ $(n+3)^3 [B_1(n+3) + B_0] - 3(n+2)^3 [B_1(n+2) + B_0] + 3(n+1)^3 [B_1(n+1) + B_0] - n^3 (B_1 n + B_0) = 3 + 5n$

ie, $B_1(n+3)^4 + B_0(n+3)^3 - 3B_1(n+2)^4 - 3B_0(n+2)^3 + 3B_1(n+1)^4 + 3B_0(n+1)^3 - B_1 n^4 - B_0 n^3 = 3 + 5n$

Equate the constant terms

$3^4 B_1 + 3^3 B_0 - 3B_1 2^4 - 3B_0 2^3 + 3B_1 + 3B_0 = 3$

$81 B_1 + 27 B_0 - 48 B_1 - 24 B_0 + 3B_1 + 3B_0 = 3$

$36 B_1 + 6 B_0 = 3$

$12 B_1 + 2B_0 = 1$ —— ②

Equate the co-efficient of 'n' on both sides.

$$B_1(n^4+12n^3+54n^2+108n+81)+$$
$$B_0(n^3+6n^2+27n+27)-3B_1(n^4+8n^3+24n^2+32n+16)$$
$$-3B_0(n^3+6n^2+12n+8)+3B_1(n^4+4n^3+6n^2+4n+1)$$
$$+3B_0(n^3+3n^2+3n+1)-B_1n^4-B_0n^3=3+5n$$

co-efficient of n is $108B_1+27B_0-96B_1-36B_0+12B_1+9B_0=5$

$$24B_1+0B_0=5$$
$$\therefore \boxed{B_1=\frac{5}{24}} \text{ --- (3)}$$

Solve (2) & (3) $\quad 12\times\frac{5}{24}+2B_0=1$

$$\frac{5}{2}+2B_0=1$$
$$2B_0=1-\frac{5}{2}$$
$$2B_0=\frac{2-5}{2}=\frac{-3}{2}$$
$$\therefore \boxed{B_0=\frac{-3}{4}}$$

$$\therefore a_n^{(p)}=n^3\left(\frac{5}{4}n-\frac{3}{4}\right)=\frac{5}{4}n^4-\frac{3}{4}n^3$$

$\therefore$ gen soln is $a_n=a_n^{(h)}+a_n^{(p)}$

$$\underline{a_n=A_0+A_1n+A_2n^2+\left(\frac{5}{4}n^4-\frac{3}{4}n^3\right)} \quad n\geq 0$$

Since the bdry conditions are not give we cannot find A_0, A_1 & A_2

Extra Problems.

4. Solve $a_{n+1} = a_n + (n+1)^3 \quad n \geq 0, \; a_0 = 0$

5. Solve $a_{n+1} = (1.05)a_n + 20{,}000\, n, \quad n \geq 1$
$a_0 = 6{,}000{,}000.$

6. $a_{n+2} - 10a_{n+1} + 21 a_n = f(n) \quad n \geq 0$

(a) $f(n) = 5$
(b) $f(n) = 3n^2 - 2$
(c) $f(n) = 7(11^n)$
(d) $f(n) = 31(r^n) \quad r \neq 3, 7$
(e) $f(n) = 6(3^n)$
(f) $f(n) = 2(3^n) - 8(9^n)$
(g) $f(n) = 4(3^n) - 3(7^n)$

Answers

4) $a_n = A + n(B_3 n^3 + B_2 n^2 + B_1 n + B_0)$

where $A = 0$, $B_3 = \frac{1}{4}$, $B_2 = \frac{1}{2}$, $B_1 = \frac{1}{4}$, $B_0 = 0$

5) $a_n = A(1.05)^n + (B_1 n + B_0)$

$A = 14{,}000{,}000$
$B_0 = -8{,}000{,}000$
$B_1 = -400{,}000$

6) $a_n = a_n^{(h)} + a_n^{(p)}$

$a_n^{(h)} = A_1(3^n) + A_2(7^n)$

A_1 & A_2 are arb. constants.

$a_n^{(p)}$

(a) $a_n^{(p)} = B.$
(b) $a_n^{(p)} = B_2 n^2 + B_1 n + B_0$
(c) $a_n^{(p)} = B \cdot 11^n$
(d) $a_n^{(p)} = B \cdot r^n$
(e) $a_n^{(p)} = n \cdot B \cdot 3^n$
(f) $a_n^{(p)} = n \cdot B_1 3^n + B_2 9^n$
(g) $a_n^{(p)} = n \cdot B_1 3^n + n \cdot B_2 7^n$

In each case find $B, B_0, B_1, B_2 \ldots$

SUMMARY to find particular Soln.

	$f(n)$	particular soln $a_n^{(P)}$
1.	$k\beta^n$	$a_n^{(P)} = n^m B \cdot \beta^n$ if β is a root of ch. eqn with multiplicity m
2.	$n^t \cdot \beta^n$	$a_n^{(P)} = n^m \left(\begin{matrix}\text{A polynomial} \\ \text{of degree } t\end{matrix}\right) \beta^n$ if β is a ch. root of multiplicity m $= n^m (B_t n^t + B_{t-1} n^{t-1} + \cdots + B_1 t + B_0)\, \beta^n$
3.	$\sin n\theta \cdot \beta^n$ $\boxed{OR}$	$a_n^{(P)} = (A \sin n\theta + B \cos n\theta)\, \beta^n$
✯	$\cos n\theta \cdot \beta^n$	$a_n^{(P)} = (A \sin n\theta + B \cos n\theta)\, \beta^n$

For Case ③ no problems are there!